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Negativity as Entanglement Degree of a Non-Hermitian Model

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Abstract We investigate non-Hermitian Hamiltonian which governs system includes two-level atoms and electromagnetical field. Using the notion of negativity, we study the degree of entanglement of a two- level atom interacting with a quantized electromagnetical field, described by the non-Hermitian Hamiltonian (Saaidi in Phys. Scr. 77:0065002, 2008). *With the help of numerical calculation for the case that the system state is pure, we show that the measurement of negativity of this system is nonzero and has a different functional with respect to negativity of the Jaynes-Cumming model* (*JCM*).

Keywords Density matrix · Entanglement · Non-Hermitian Hamiltonian

1 Introduction

The question of quantum inseparability and entanglement of mixed state has attracted much attention presently. Since the famous Einstein, Podolsky and Rosen [2] and Schrödinger [3] papers quantum entanglement still remains one of the most striking implication of quantum formalism. It has been recognized that entanglement provides a fundamental potential resource for communication and information processing [4–6]. In practice, one usually deals with entanglement represented by mixed state of composite system. A pure quantum state of two or more subsystems is said to be entangled if it is not a product of states of each subsystems. On the other hand, bipartite mixed state ρ is supposed to represent entanglement if it can not be written as [7].

$$\rho = \sum P_i \rho_i^A \otimes \rho_i^B, \quad P_i > 0, \ \sum P_i = 1. \tag{1}$$

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Here ρ_i^A and ρ_i^B are state for the two subsystems. Peres [8] has shown that a necessary condition for separability is that a matrix, obtained by partial transposition of ρ, has only non negative eigenvalues. Horodecki et al. [9] have shown that this condition is sufficient for separability of composite system only when the dimension of the composite Hilbert space is $2 \otimes 3$ or $2 \otimes 2$. Also, it has been shown that if a mixed state can be distilled to the singlet form, it must violate partial transposition criterion [10]. Many efforts have been devoted to quantify entanglement, particularly for mixed states of a bipartite system, and number of measures have been proposed, such as entanglement of formation (*EoF*) [6, 11], relative entropy of entanglement (*REE*) [12, 13], and negativity [8, 9, 14–16].

Quantum optics provides the ideal area to deal with the interaction of radiation and matter. A numerous works have been devoted to the measurement of entangled state between atom and electromagnetic field (EM). Indeed, the most important and well known concept which have been introduced so far in the quantum optic category is the two level atom. Some investigation have been done about the measurement of entangled state between atom and EM field with the base of the two level atoms, e.g. the entangled atom field state in all time [17], the entanglement of the Jaynes-Cumming model (JCM) with the atom initially in a mixed state and field in a squeezed state [18], the entanglement of the JCM with the atom initially in mixed state and field in a coherent state [19], the entanglement of the JCM with the atom initially in mixed state and field in an arbitrary thermal state [20]. Some another investigation about entanglement of JCM is done in [21–29].

In the most of studies on the entanglement of JCM, index of correlation (quantum mutual entropy) is adapted to measure degree of entanglement of atom and the field [18, 19, 24, 30–33]. Also the authors of [34] have studied the negativity as the entanglement degree of the JCM. They have shown that the whole density matrix has rank two and supported at must on $\mathbb{C}^2 \otimes \mathbb{C}^4$ space.

In [1], *we introduced and studied a non-Hermitian Hamiltonian which describe the interaction between tow-level atom with EM field. We have shown that the physical results of that model have an excellent agreement with the physical results of JCM.* In this work, we shall turn our attention to concentrate on the entanglement of that model. We will study the negativity as the entanglement degree of it. We suppose that the atom is prepared, initially, in mixed state and the field is in a pure state. It is seen that the whole density matrix, which is a σ_z-pseudo Hermitian operator, has rank two and supported at most on $\mathbb{C}^2 \otimes \mathbb{C}^4$ space. Where as all positive partial transpose rank two bipartite density matrices are separable [35, 36], thus the negativity fully captures the entanglement properties of our model.

2 Preliminary

Let H be a linear operator acting in a Hilbert space and η be a linear, Hermitian, invertible operator. Then the η-pseudo-Hermitian of H is defined as [37–41]

$$H = \eta^{-1} H^{\dagger} \eta. \tag{2}$$

The Hermitian indefinite inner product defined by η as

$$\langle\langle \psi_1 \mid \psi_2 \rangle\rangle_\eta = \langle \psi_1 \mid \eta \psi_2 \rangle. \tag{3}$$

If H be η-pseudo-Hermitian this inner product is invariant under the time-translation generated by the H and also under this inner product the Hamiltonian is Hermitian, i.e.;

$$\langle\langle \psi_1 \mid H \psi_2 \rangle\rangle_\eta = \langle\langle H \psi_1 \mid \psi_2 \rangle\rangle_\eta. \tag{4}$$

The pseudo-Hermitian Hamiltonian has a complete biorthonormal set of eigenvectors $\{|\psi_n^i\rangle, |\phi_m^j\rangle\}$ which satisfy the following relation

$$
\begin{aligned}
H_{tot}|\psi_n^i\rangle &= E_n|\psi_n^i\rangle,\\
H_{tot}^{\dagger}|\phi_n^i\rangle &= E_n^*|\phi_n^i\rangle,\\
\langle\psi_n^i|\phi_m^j\rangle &= \delta^{ij}\delta_{nm},\\
\sum_n\sum_i |\psi_n^i\rangle\langle\phi_n^i| &= \textstyle\sum_n\sum_i |\phi_n^i\rangle\langle\psi_n^i| = I.
\end{aligned}
\tag{5}
$$

Here n and i are the spectral and degeneracy label respectively, δ_{nm} denote the Kronecker delta function, and 1 is the identity operator [37].

3 The Non-Hermitian Model

In this section we consider the Hamiltonian with Non-Hermitian interaction as,

$$
H_{tot} = \frac{\hbar\omega}{2}\sigma_z + a^{\dagger}a\hbar\nu + i\hbar\lambda(\sigma^+a + \sigma^-a^{\dagger}).
\tag{6}
$$

It is easily seen that this Hamiltonian, (6), *satisfy the relation* (2) *with respect to* $\eta \equiv \sigma_z$. Let the two possible energy levels of the atom be denoted by $\pm\frac{\hbar\omega}{2}$, and the corresponding states by $|s\rangle$, $(s = e, g)$. Similarly, the number of quanta (photon) in the field oscillator will be n, and the corresponding state of the field by $|n\rangle$ $(n = 0, 1, 2, \ldots)$. Obviously $\sigma_z|s\rangle = \xi_s|s\rangle$, $(\xi_e = 1, \xi_g = -1)$. We denote the state of the total Hamiltonian by $|s, n\rangle$ and it is well known that the projection operators $\sigma^{\pm}$ have the following usual properties when they act on the states $|s, n\rangle$, $\sigma^+|e, n\rangle = 0$, $\sigma^-|g, n\rangle = 0$, $\sigma^+|g, n\rangle = |e, n\rangle$, $\sigma^-|e, n\rangle = |g, n\rangle$. Although, $H_{tot}|g, 0\rangle = -\frac{\hbar\omega}{2}|g, 0\rangle$, it is easily seen that $|g, 0\rangle$ is a ground state of the Hamiltonian. By applying H_{tot} on $|e, n\rangle$ and $|g, n+1\rangle$, we have

$$
\begin{aligned}
H_{tot}|e, n\rangle &= \hbar\left(n\nu + \frac{\omega}{2}\right)|e, n\rangle + i\hbar\lambda\sqrt{n+1}|g, n+1\rangle,\\
H_{tot}|g, n+1\rangle &= \hbar\left((n+1)\nu - \frac{\omega}{2}\right)|g, n+1\rangle + i\hbar\lambda\sqrt{n+1}|e, n\rangle.
\end{aligned}
\tag{7}
$$

So that the Hamiltonian matrix in the $|s, n\rangle$ basis is given by

$$
H_{tot} = \begin{bmatrix} \hbar(n\nu + \frac{\omega}{2}) & i\hbar\lambda\sqrt{n+1} \\ i\hbar\lambda\sqrt{n+1} & \hbar((n+1)\nu - \frac{\omega}{2}) \end{bmatrix}.
\tag{8}
$$

The eigenvalues of this Hamiltonian matrix are given by

$$
E_n^{(\mp)} = \left(n + \frac{1}{2}\right)\hbar\nu \mp \Omega_n,
\tag{9}
$$

where

$$
\Omega_n = \sqrt{\left(\frac{\Delta}{2}\right)^2 - \lambda^2(n+1)}.
$$

Note, these eigenvalue are real provided

$$\left(\frac{\Delta}{2}\right)^2 \geq \lambda^2(n+1). \tag{10}$$

So, by defining $\lambda^2(n+1) = (\frac{\Delta}{2})^2 \sin^2(2\theta_n)$ and then the two eigenstates corresponding with two eigenvalues are

$$\begin{aligned} |\psi_n^{(+)}\rangle &= |A_{n+1}|[\sin(\theta_n)|e,n\rangle - i\cos(\theta_n)|g,n+1\rangle], \\ |\psi_n^{(-)}\rangle &= |B_{n+1}|[\cos(\theta_n)|e,n\rangle - i\sin(\theta_n)|g,n+1\rangle], \end{aligned} \tag{11}$$

where A_{n+1} and B_{n+1} are normalization constants,

$$|A_{n+1}|^2 = -|B_{n+1}|^2 = \frac{1}{\sin^2(\theta_n) - \cos^2(\theta_n)}. \tag{12}$$

It is easily seen that the eigenstates in (11) satisfy the pseudo inner product, (3), with respect to σ_z, as

$$\begin{aligned} \langle\langle \psi_n^{(i)} \mid \psi_m^{(j)}\rangle\rangle_{\sigma_z} &= \langle \psi_n^{(i)} \mid \sigma_z \psi_m^{(j)}\rangle, \\ &= \langle \psi_n^{(i)} \mid \phi_m^{(j)}\rangle = \delta_{nm}\delta^{ij}. \end{aligned} \tag{13}$$

Here $|\phi_m^{(i)}\rangle = \sigma_z|\psi_m^{(i)}\rangle$ $(i = +, -)$ and $\langle . \mid .\rangle$ is the original inner product. One can find that $|\phi_n^{(i)}\rangle$'s are the eigenkets of $H_{tot}^\dagger$ with eigenvalue $E_n^{(i)}$. Therefore, the Hamiltonian, H_{tot}, is called σ_z-pseudo Hermitian Hamiltonian and has a complete set of biorthonormal eigenkets $\{|\psi_n^{(i)}\rangle, |\phi_n^{(i)}\rangle\}$ which satisfy the set of equations (5), and also the Hamiltonian H_{tot} is Hermitian with respect to the pseudo-inner product, (3) as

$$\begin{aligned} \langle\langle \psi_1 \mid H\psi_2\rangle\rangle_{\sigma_z} &= \langle \psi_1 \mid \sigma_z H\psi_2\rangle = \langle \psi_1 \mid H^\dagger \sigma_z \psi_2\rangle, \\ &= \langle H\psi_1 \mid \sigma_z\psi_2\rangle = \langle\langle H\psi_1 \mid \psi_2\rangle\rangle_{\sigma_z}. \end{aligned} \tag{14}$$

4 The Time Evolution of the System

In order to study the entanglement properties of the σ_z-pseudo-Hermitian Hamiltonian, (6), let us suppose that, initially at $t = 0$, the system is found in the product state as

$$\rho(0) = \rho^A(0) \otimes \rho^F(0). \tag{15}$$

Where $\rho^A(0)$ is the initial state of the atom which is a general mixed state with the diagonal representation. Also, $\rho^F(0)$, the initial state of the filed, is a general coherent pure state

$$\rho^F(0) = |\eta\rangle\langle\eta|, \quad |\eta\rangle = \sum_{n=0}^{\infty} b_n|n\rangle, \tag{16}$$

where the coefficients $b_n = \langle n|\eta\rangle$ are such that the state is normalized, i.e. $\sum_{n=0}^{\infty}|b_n|^2 = 1$. Therefore, we obtain (15) as

$$\rho(0) = \omega_g|g,\eta\rangle\langle\widetilde{g,\eta}| + \omega_e|e,\eta\rangle\langle\widetilde{e,\eta}|, \qquad \omega_g + \omega_e = 1, \tag{17}$$

where $|\widetilde{e,\eta}\rangle = \sigma|e,\eta\rangle$. We can obtain the final state of the system as

$$\rho(t) = \sum_{s=e,g} \omega_s |\Psi_s(t)\rangle\langle\Phi_s(t)|, \tag{18}$$

$$|\Phi_s(t)\rangle = \sigma_z|\Psi_s(t)\rangle = e^{\frac{-iH^\dagger t}{\hbar}}|\widetilde{s,\eta}\rangle, \tag{19}$$

$$|\Psi_s(t)\rangle = e^{\frac{-iHt}{\hbar}}|s,\eta\rangle. \tag{20}$$

Thus we have

$$|\Psi_s(t)\rangle = \sum_n^\infty \sum_{i=+,-} e^{\frac{-iE_n^{(i)}t}{\hbar}} |\psi_n^{(i)}\rangle\langle\phi_n^{(i)}|s,\eta\rangle, \tag{21}$$

$$|\Phi_s(t)\rangle = \sum_n^\infty \sum_{i=+,-} e^{\frac{-iE_n^{(i)}t}{\hbar}} |\phi_n^{(i)}\rangle\langle\psi_n^{(i)}|\widetilde{s,\eta}\rangle. \tag{22}$$

Then one can obtain $|\Psi_s(t)\rangle$ and $|\Phi_s(t)\rangle$, for $s = e, g$ as follows:

$$|\Psi_g(t)\rangle = |g\rangle \otimes |\chi_1(t)\rangle + |e\rangle \otimes |\chi_2(t)\rangle, \tag{23}$$

$$|\Psi_e(t)\rangle = |g\rangle \otimes |\chi_3(t)\rangle + |e\rangle \otimes |\chi_4(t)\rangle, \tag{24}$$

$$|\Phi_g(t)\rangle = |g\rangle \otimes |\chi_1'(t)\rangle + |e\rangle \otimes |\chi_2'(t)\rangle, \tag{25}$$

$$|\Phi_e(t)\rangle = |g\rangle \otimes |\chi_3'(t)\rangle + |e\rangle \otimes |\chi_4'(t)\rangle, \tag{26}$$

where the non-normalized vector $|\chi_\alpha(t)\rangle$, and $|\chi_\alpha'(t)\rangle$, $(\alpha = 1,\dots,4)$ are given by

$$|\chi_1(t)\rangle = i\sum_{n=0}^\infty e^{-i(n-\frac{1}{2})\nu t} b_n |A_n|^2 [\sin\Omega_{n-1}t + i\cos\Omega_{n-1}t\cos 2\theta_{n-1}]|n\rangle, \tag{27}$$

$$|\chi_2(t)\rangle = -\sum_{n=0}^\infty e^{-i(n+\frac{1}{2})\nu t} b_{n+1} |A_{n+1}|^2 [\sin 2\theta_n \sin\Omega_n t]|n\rangle, \tag{28}$$

$$|\chi_3(t)\rangle = -\sum_{n=0}^\infty e^{-i(n+\frac{1}{2})\nu t} b_n |A_{n+1}|^2 [\sin 2\theta_n \sin\Omega_n t]|n+1\rangle, \tag{29}$$

$$|\chi_4(t)\rangle = -\sum_{n=0}^\infty e^{-i(n+\frac{1}{2})\nu t} b_n |A_{n+1}|^2 [\cos\Omega_n t\cos 2\theta_n + i\sin\Omega_n t]|n\rangle, \tag{30}$$

$$|\chi_1'(t)\rangle = i\sum_{n=0}^\infty e^{-i(n+\frac{1}{2})\nu t} b_{n+1} |A_{n+1}|^2 [\sin\Omega_n t + i\cos\Omega_n t\cos 2\theta_n]|n+1\rangle, \tag{31}$$

$$|\chi_2'(t)\rangle = \sum_{n=0}^\infty e^{-i(n+\frac{1}{2})\nu t} b_{n+1} |A_{n+1}|^2 [\sin 2\theta_n \sin\Omega_n t]|n\rangle, \tag{32}$$

$$|\chi_3'(t)\rangle = \sum_{n=0}^\infty e^{-i(n+\frac{1}{2})\nu t} b_n |A_{n+1}|^2 [\sin 2\theta_n \sin\Omega_n t]|n+1\rangle, \tag{33}$$

$$|\chi_4'(t)\rangle = -\sum_{n=0}^{\infty} e^{-i(n+\frac{1}{2})\nu t} b_n |A_{n+1}|^2 [\cos\Omega_n t \cos 2\theta_n + i\sin\Omega_n t]|n\rangle. \tag{34}$$

Alternatively, given an orthonormal basis $\{|e_1\rangle \equiv |e\rangle, |e_2\rangle \equiv |g\rangle\} \in \mathbb{C}^2$ for the atomic Hilbert space, we can represent the density matrix $\rho(t)$ as

$$\rho(t) = \begin{pmatrix} A(t) & C(t) \\ C'(t) & B(t) \end{pmatrix}, \tag{35}$$

where $A(t)$, $B(t)$, $C(t)$ and $C'(t)$, operators acting on the filed Hilbert space, are defined by

$$A(t) = \omega_g |\chi_2(t)\rangle\langle\chi_2'(t)| + \omega_e |\chi_4(t)\rangle\langle\chi_4'(t)|, \tag{36}$$

$$B(t) = \omega_g |\chi_1(t)\rangle\langle\chi_1'(t)| + \omega_e |\chi_3(t)\rangle\langle\chi_3'(t)|, \tag{37}$$

$$C(t) = \omega_g |\chi_2(t)\rangle\langle\chi_1'(t)| + \omega_e |\chi_4(t)\rangle\langle\chi_3'(t)|, \tag{38}$$

$$C'(t) = \omega_g |\chi_1(t)\rangle\langle\chi_2'(t)| + \omega_e |\chi_3(t)\rangle\langle\chi_4'(t)|. \tag{39}$$

Therefore, in the basis $\{|e_1\rangle, |e_2\rangle\}$, the atomic density matrix has the following matrix elements

$$\begin{aligned}\rho_{11}^A = \sum_{n=0}^{\infty} |A_{n+1}|^4 \Big[& -\omega_g |b_{n+1}|^2 (\sin^2\Omega_n t \sin^2 2\theta_n) \\ & + \omega_e |b_n|^2 (\cos^2\Omega_n t \cos^2 2\theta_n + \sin^2\Omega_n t)\Big],\end{aligned} \tag{40}$$

$$\begin{aligned}\rho_{22}^A = \sum_{n=0}^{\infty} |A_{n+1}|^4 \Big[& \omega_g |b_{n+1}|^2 (\cos^2\Omega_n t \cos^2 2\theta_n + \sin^2\Omega_n t) \\ & - \omega_e |b_n|^2 (\sin^2\Omega_n t \sin^2 2\theta_n)\Big],\end{aligned} \tag{41}$$

$$\begin{aligned}\rho_{12}^A = \sum_{n=0}^{\infty} & e^{-i\nu t} |A_n|^2 |A_{n+1}|^2 \\ & \times [\omega_g b_{n+1} b_n^* (\sin\Omega_n t \sin 2\theta_n)(i\sin\Omega_{n-1} t + \cos\Omega_{n-1} t \cos 2\theta_{n-1}) \\ & + \omega_e b_n b_{n-1}^* (\sin\Omega_{n-1} t \sin 2\theta_{n-1})(-i\sin\Omega_n t + \cos\Omega_n t \cos 2\theta_n)],\end{aligned} \tag{42}$$

$$\begin{aligned}\rho_{21}^A = \sum_{n=0}^{\infty} & e^{i\nu t} |A_n|^2 |A_{n+1}|^2 \\ & \times \Big[\omega_g b_{n+1}^* b_n (\sin\Omega_n t \sin 2\theta_n)(i\sin\Omega_{n-1} t - \cos\Omega_{n-1} t \cos 2\theta_{n-1}) \\ & + \omega_e b_n^* b_{n-1} (\sin\Omega_{n-1} t \sin 2\theta_{n-1})(-i\sin\Omega_n t + \cos\Omega_n t \cos 2\theta_n)\Big].\end{aligned} \tag{43}$$

It is clearly seen that $\rho(t) = \sigma_z^{-1}\rho^\dagger(t)\sigma_z$, so that we defined a density matrix as $\rho'(t) = \sigma_z\rho(t)$ which is a Hermitian operator. Therefore $\rho_{22}'^A = 1 + \rho_{11}'^A$ and $(\rho_{12}'^A)^* = \rho_{21}'^A$. These relation is equivalent to $(\rho_{12}^A)^* = -\rho_{21}^A$, and $\rho_{22}^A = 1 - \rho_{11}^A$. Then

$$\begin{aligned}\rho^F(t) &= A(t) + B(t), \\ &= \omega_g(|\chi_1(t)\rangle\langle\chi_1'(t)| + |\chi_2(t)\rangle\langle\chi_2'(t)|) \\ &\quad + \omega_e(|\chi_3(t)\rangle\langle\chi_3'(t)| + |\chi_4(t)\rangle\langle\chi_4'(t)|). \end{aligned} \tag{44}$$

5 Negativity

For quantify the entanglement of the final state (35), using the concept of the negativity defined by [42]

$$N(\rho) \equiv \frac{\|\rho^{T_1}\|_1 - 1}{2}. \tag{45}$$

Where ρ^{T_1} is the matrix obtained by partially transposing the density matrix ρ whit respect to the first system, and $\|\rho^{T_1}\|_1$ is the trace class norm of the operator ρ^{T_1}. The trace class norm of any trace class operator A is defined by $\|A\|_1 = \mathrm{Tr}\sqrt{A^\dagger A}$, which reduces to the some of the absolute value of the eigenvalues of A, when A is Hermitian. So that we use $\rho'(t)$ which

$$\|\rho'\|_1 = \sqrt{\rho'^\dagger \rho'} = \sqrt{\rho^\dagger \sigma^\dagger \sigma \rho} = \sqrt{\rho^\dagger \rho} = \|\rho\|_1, \tag{46}$$

therefore

$$\|\rho^{T_1}\| = \sum_i |\mu_i| = \sum_i \mu_i - 2\sum_i \mu_i^{neg} = 1 - 2\sum_i \mu_i^{neg}, \tag{47}$$

where μ_i and μ_i^{neg} are, respectively, the eigenvalues and the negative eigenvalues of ρ^{T_1}. In the last step, we used also the fact that $\mathrm{Tr}\,\rho^{T_1} = \mathrm{Tr}\,\rho = 1$. The partial transposition of $\rho(t)$ with respect to the atom in the basis $|e_1\rangle, |e_2\rangle \in \mathbb{C}^2$ is defined by

$$\rho^{T_1}(t) = \sum_{i,j=1}^{2} \langle e_i|\rho(t)|e_j\rangle |e_j\rangle\langle e_i|, \tag{48}$$

where, using the representation (35), can be written in matrix form as

$$\rho^{T_1}(t) = \begin{pmatrix} A(t) & C'^\dagger(t) \\ C^\dagger(t) & B(t) \end{pmatrix}. \tag{49}$$

We can calculate the eigenvalues of $\rho^{T_1}(t)$ by expanding the field operators of (35) in number states $\{|n\rangle\}_{n=0}^{n=\infty}$ as

$$\begin{aligned} A_{nm} &= e^{-i(m-n)\nu t}|A_m|^2|A_n|^2 \\ &\quad \times \Big[-\omega_e b_{m+1} b_{n+1}^* (\sin\Omega_m t \sin\Omega_n t \sin 2\theta_m \sin 2\theta_n) \\ &\quad + \omega_g b_m b_n^* (\cos 2\theta_m \cos\Omega_m t + i\sin\Omega_m t)(\cos 2\theta_n \cos\Omega_n t - i\sin\Omega_n t)\Big], \end{aligned} \tag{50}$$

$$B_{mn}=e^{-i(m-n)\nu t}|A_m|^2|A_n|^2$$
$$\times\Big[\omega_e b_{m+1}b^*_{n+1}(i\sin\Omega_m t-\cos\Omega_m t\cos 2\theta_m)(i\sin\Omega_n t+\cos\Omega_n t\cos 2\theta_n)$$
$$+\omega_g b_m b^*_n(\sin\Omega_m t\sin\Omega_n t\sin 2\theta_m\sin 2\theta_n)\Big], \tag{51}$$

$$C_{mn}=e^{-i(m-n+1)\nu t}|A_m|^2|A_{n-1}|^2$$
$$\times\Big[\omega_e b_{m+1}b^*_{n}(\sin\Omega_m t\sin 2\theta_m)(i\sin\Omega_{n-1}t+\cos\Omega_{n-1}t\cos 2\theta_{n-1})$$
$$+\omega_g b^*_m b_{n-1}(\sin\Omega_{n-1}t\sin 2\theta_{n-1})(i\sin\Omega_m t+\cos\Omega_m t\cos 2\theta_m)\Big], \tag{52}$$

$$C'mn=e^{i(m-n+1)\nu t}|A_m|^2|A_{n-1}|^2$$
$$\times\Big[\omega_e b^*_{m+1}b_{n}(\sin\Omega_m t\sin 2\theta_m)(-i\sin\Omega_{n-1}t+\cos\Omega_{n-1}t\cos 2\theta_{n-1})$$
$$+\omega_g b^*_m b_{n-1}(\sin\Omega_{n-1}t\sin 2\theta_{n-1})(-i\sin\Omega_m t+\cos\Omega_m t\cos 2\theta_m)\Big]. \tag{53}$$

It is seen that this representation gives an infinite dimensional matrix for $\rho^{T_1}(t)$ which should be solved for its eigenvalues. The authors of [34] have shown that, for large values of m and n the above matrix elements rapidly tends to zero. For example for $|\alpha|=\sqrt{5}$ we find that $|b_n|^2$ is less then 10^{-12}, for $n>25$. It has been shown that if the rank of a density matrix is equal to two, then the positive partial transpose (PPT) is a necessary and sufficient condition for separability [7]. So that, for this model, we can calculate the eigenvalues of $\rho^{T_1}(t)$ by a truncated finite dimensional matrix representation of $\rho^{T_1}(t)$ for some n. It is clearly seen that, we have only at most eight nonzero eigenvalue for $\rho^{T_1}(t)$. Therefore according to (18), the final state of our model is a rank two density matrix supported at most on $\mathbb{C}^2\otimes\mathbb{C}^4$. This mean that the positive partial transpose condition is a necessary and sufficient condition for separability of $\rho(t)$. Consequently, negativity of a the partial transpose fully captures the entanglement of $\rho(t)$, i.e.; $\rho(t)$ is separable if $N(\rho(t))=0$.

6 Numerical Results

Although rank $\rho^{T_1}(0)=\text{rank}(\rho(0))=2$, but, due to the time evolution of the state and generation of entanglement, the rank of $\rho^{T_1}(t)$ is no longer equal to 2 and it has been shown that it may will be in general $2\le\text{rank}\,\rho^{T_1}(t)\le 8$ for this model [34]. Therefore in this case, we can not calculate the negativity analytically and numerical calculation is required. We assume that the initial state of the field is a coherent state $|\alpha\rangle=\Sigma_{n=0}^{\infty}b_n|n\rangle$ with $b_n=\frac{\alpha^n}{\sqrt{n!}}e^{\frac{-|\alpha|^2}{2}}$. Our numerical results of the negativity are shown in Figs. 1, 2 and 3. Figure 1 shows the time evaluation of the negativity when the atom is initially in the excited state, i.e.; $\omega_g=0$.

In this case it is seen that the oscillation of the negativity for $\alpha^2=7$ and $\alpha^2=5$ are nearly similar but the value of the negativity for $\alpha^2=7$ is bigger then $\alpha^2=5$. Figures 2, 3 show the time evaluation of the negativity for $\omega_e=\frac{1}{2}$. It is clearly apparent that these figures show different pattern of negativity. It is obviously seen that the time evaluation in the subfigures a and b in Fig. 2 is similar and the amplitude of the osculation and the value of the negativity for the case $\alpha^2=7$ is bigger than $\alpha^2=5$.

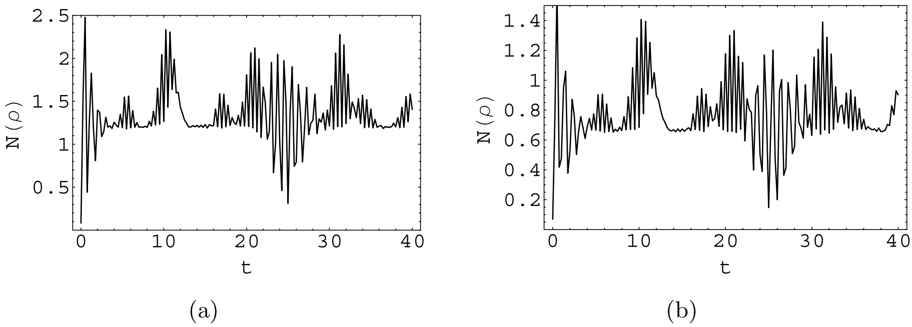

Fig. 1 **a** The time evaluation of the negativity for $\alpha^2 = 7$, $\lambda = 1.5$, $\nu = 1$, $\Delta = 20$, $\omega_g = 0$. **b** The time evaluation of the negativity for $\alpha^2 = 5$, $\lambda = 1.5$, $\nu = 1$, $\Delta = 20$, $\omega_g = 0$

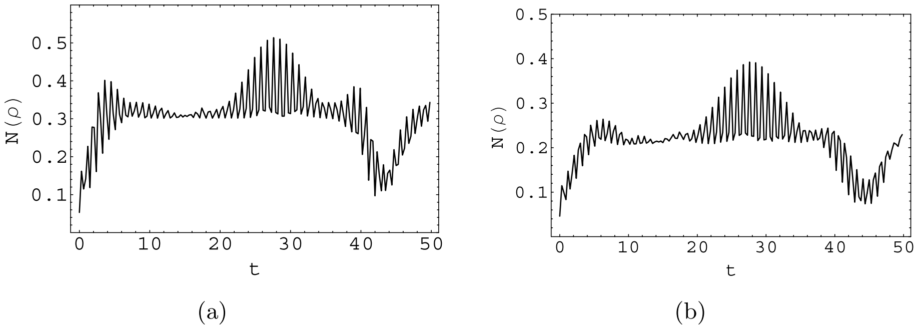

Fig. 2 **a** The time evaluation of the negativity for $\alpha^2 = 7$, $\lambda = 1$, $\nu = 1$, $\Delta = 15$, $\omega_e = 0.5$. **b** The time evaluation of the negativity for $\alpha^2 = 5$, $\lambda = 1$, $\nu = 1$, $\Delta = 15$, $\omega_e = 0.5$

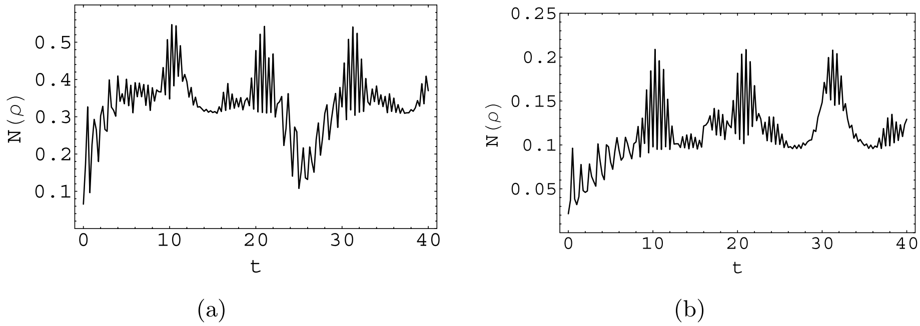

Fig. 3 **a** The time evaluation of the negativity for $\alpha^2 = 5$, $\lambda = 1.5$, $\nu = 1$, $\Delta = 20$, $\omega_e = 0.5$. **b** The time evaluation of the negativity for $\alpha^2 = 5$, $\lambda = 1$, $\nu = 1$, $\Delta = 20$, $\omega_e = 0.5$

In Fig. 3, we have drawn the time evaluation of the negativity for different coupling constant. It is seen that by increasing the coupling constant the amplitude of oscillation and value of the negativity is increased.

7 Conclusion

In this paper we have used the negativity in order to quantify the entanglement degree of the our model which represent in [1]. We have assume that the initial state of the atom is a general mixed state, and the field in initially in coherent state. It is seen that the density operator $\rho(t)$ is a σ_z-pseudo-Hermitian operator, $\rho(t) = \sigma_z \rho^\dagger(t) \sigma_z^{-1}$ and acts on $\mathbb{C}^2 \otimes \mathbb{C}^4$ space which $1 \leq N \leq 4$. In this case the negativity fully captures the entanglement properties of the model. With the help of numerical calculation for the case that the system state is pure, we shown that the measurement of negativity of this system has the different functional with respect to JCM. It is investigated that by increasing the field intensity, the negativity shows the same functionality, up to a difference in the amplitude oscillation and the value of the negativity is increased. The effect of the coupling constant is also examined, and it is shown that by increasing the coupling constant the negativity is increased.

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